"Connecting Two and Four Dimensional Physics in a Supersymmetric World"

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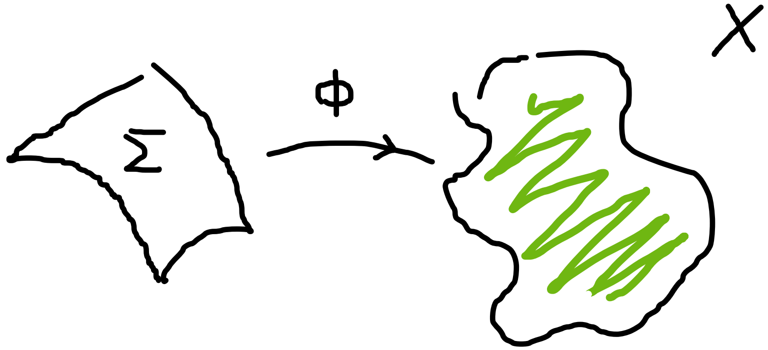
Based on 1104 3021, 1205 4230 with Nick Dorey, Sungjay Lee, Peng Zhao (Cambridge) and Tim Hollowood (Swansea)

+ 1303 4237 with Annamaria Sinkovics

Two dimensional σ-model have served as a powerful paradigm in helping us understanding Four dimensional non-Abelian theories

They are known to share many important features

- Asymptotic Freedom
- Generation of mass gap
- Confinement
- Chiral Symmetry Breaking
- Large N-expansion
- Instantons
- Anomalies / Current Algebra



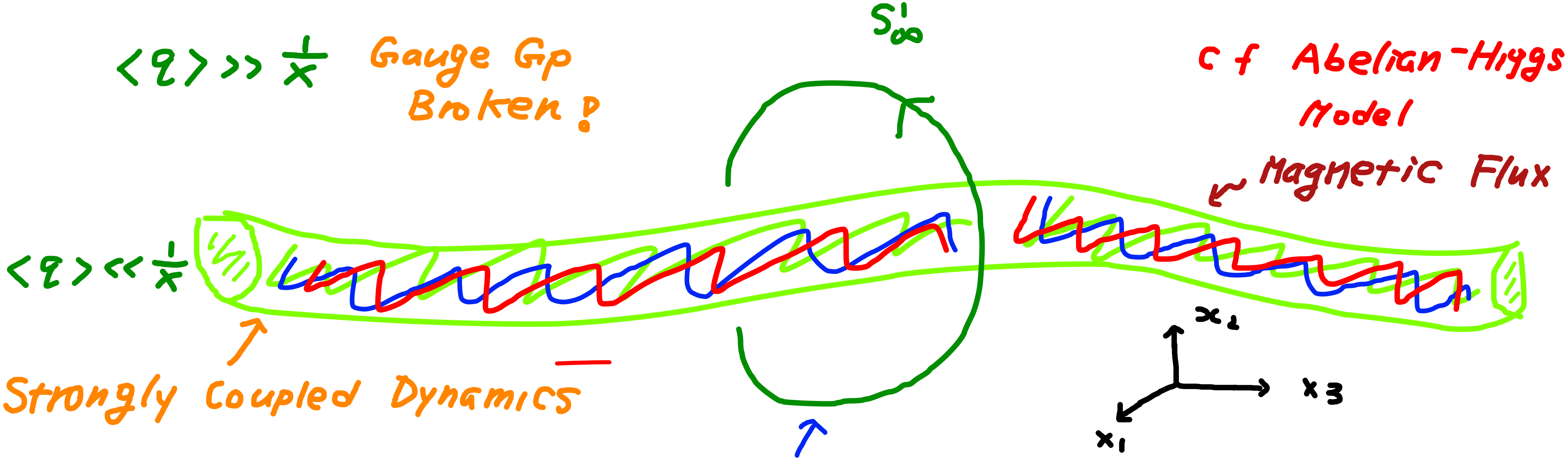
Gross-Neveu, Witten, D'Adda-Luscher-Di Vecchia etc

But 2d σ-models are "a bit nicer", exhibiting Striking features: such as Bosonization, Infinite Conserved Charges, important for us "Connection with Integrable Systems"

The qualitative connections between 2 dim σ-models and 4 dim Non-Abelian Gauge Theories can be made "Quantitative"

"Solitonic Vortex Strings Precisely Provides Such Connection"

The Basic Idea is, we consider 4 dim Non-Abelian Theory with some quarks q (Scalar), we go to "Higgs Phase" $\langle q \rangle \neq 0$



Non-Trivial Winding of $q(\vec{x})$

$$2\pi K = i \oint_{S^1_\infty} d \log q$$

More Concretely, we Consider 4 dim Theory with $U(N_c)$ Gauge Gp and N_f flavors (+ SUSY generalization) (U(1) Abelian Higgs)

$$\mathcal{L} = \frac{1}{4e^2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{N_f} |Dq_i|^2 - \frac{e^2}{2}\mathrm{Tr}\Big(\sum_{i=1}^{N_f}|q_i|^2 - \underset{\text{FI-Parameter}}{v^2}\Big)^2$$

(+ Fermions for SUSY)

$q^\alpha_i \quad \alpha = 1, \cdots N_c, \quad i = 1, \cdots N_f$

Vacuum $\quad \langle q^\alpha{}_i \rangle = v\,\delta^\alpha_i \quad , \quad \underbrace{U(N_c)\times SU(N_f) \to}_{\text{Overall } U(1) \text{ is broken by the vev}} \overbrace{S[U(N_c)_d \times U(N_f - N_c)]}^{N_c = N_f \text{ "Color-Flavor Locking"}}$

Mass Gap $\quad m_q = m_g \sim ev$

$\rightsquigarrow$ Solitonic Vortex String Solution exists, labeled by

$$\pi_1\left(\frac{U(N_c)\times SU(N_f)}{S[U(N_c)_d\times U(N_f - N_c)]}\right) = \mathbb{Z} \quad \leftarrow \text{Vortex Number}$$

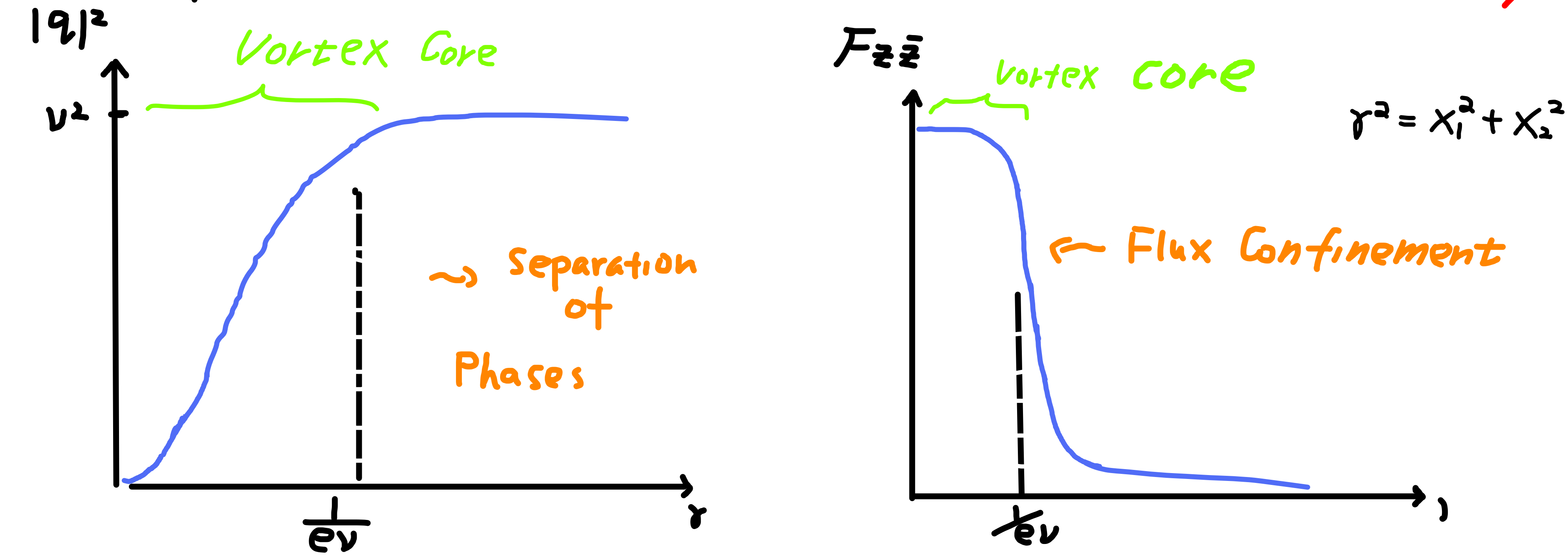
To find vortex solution, use Bogomolny Completing the square trick

Vortex Equation (in x^0-x^3)

$$(F_{z\bar{z}})^{\alpha}{}_{\beta} = e^2\left(\sum_{i=1}^{N_f} q^{\alpha}_{i}\, q^{\dagger}_{i\,\beta} - v^2 \delta^{\alpha}_{\beta}\right)$$

$$D_{\bar{z}}\, q = 0 \qquad (z = x_1 + i x_2)$$

The analytic Solutions unknown, but it can be solved numerically



The vortex world volume dynamics (Two Dimensional) "Probes" the 4 dim Strong Coupling Dynamics ?

The low energy vortex dynamics is given by "Moduli Space Approximation"
(Manton)

Moduli space metric g_{IJ} can be obtained from

$$g_{IJ} = \mathrm{Tr} \int d^2 z \; \frac{1}{e^2} \delta_I A_z \delta_J A_{\bar{z}} + \sum_{i=1}^{N_f} \delta_I q_i^\alpha \delta_J q_\alpha^{\dagger i} + h.c$$

$(\delta_I A, \delta q)$ are so-called "Zero Modes" (Bosonic) (Linearized Fluctuations)

We can also identify moduli space by considering different ways of embedding vortex solutions For $k=1$, $N_f = N_c$

Widely Studied

$$\mathcal{M}_{1,N_c,N_c} = \frac{SU(N_c)_d}{U(1)\times SU(N_c-1)} \cong \mathbb{CP}^{N_c-1}$$

"The low energy dynamics is given by 2 dim σ-model with target space being $\mathbb{CP}^{N_c-1}$ The size/Kähler is proportional to $\frac{1}{e^2}$"

More general configuration, arbitrary k, N_c and N_f etc we can obtain the moduli space $\mathcal{M}_{k,N_c,N_f}$ via D-brane construction
(Hanany-Tong, Dorey, Hollowood etc)

- If the 4 dim Gauge Theory also has "fermions", they also descend into 2 dim Vortex world volume as "fermionic zero modes"

$\Rightarrow$ Combined with "bosonic zero models", We can have "Supersymmetry" (in 2 dim)

- Here we are most interested in $\mathcal{N}=(2,2)$ Supersymmetric σ-models

So far the discussion is classical, for their "quantum dynamics", its more powerful to rewrite them as "Gauge Theories" (Witten), sometimes referred as "Gauged Linear Sigma Model (GLSM)"

- Various Aspects of $\mathcal{N}=(2,2)$ GLSM have been studied, (Dorey, Shifman-Yung, Hanany-Tong), in particular the Soliton Spectrum bears close resemblence to the BPS Soliton Spectrum of famous "4 dim $\mathcal{N}=2$ Supersymmetric Gauge Theories (Seiberg-Witten)"

Coupling

The Particular $\mathcal{N}=(2,2)$ GLSM we focus on has $G=U(K)$ g

$+$ N_c fundamental Chiral Multiplets of (twisted) masses $(M_1, M_2 \quad M_{N_c})$,

$+$ N_c anti-fundamental Chiral Multiplets of (twisted) masses $(\tilde{M}_1, \tilde{M}_2 \cdots \tilde{M}_{N_c})$,

$+$ 1 adjoint Chiral Multiplet of (twisted) mass ϵ

We also add Fayet-Iliopoulos term (FI) r + 2d theta angle θ_{2d}

and combine to form $\tau = ir + \frac{\theta_{2d}}{2\pi}$ ⤳ Coupled through Field Strength

Interesting low energy dynamics or "quantum vacua" is governed

by an "Exact twisted Superpotential $W_{eff}(\lambda)$" (λ ~ twisted chiral mult)

(Witten, Cecotti-Vafa)

$$\mathcal{L}_{eff} = \underbrace{\int d^2\theta\, W_{eff}(\lambda) + \int d^2\bar{\theta}\, \overline{W}_{eff}(\bar{\lambda})}_{\text{Potential}} + \int d^4\theta\, K(\lambda, \bar{\lambda})$$

↑ kinectic terms

$W_{eff}(\lambda)$ ~ obtained from integrating out massive matters

Explicit Form of $W_{eff}(\Sigma)$ is

"Generating Function for Quantum Integrable System"

$$W_{eff}(\lambda) = 2\pi i \tau \sum_{j=1}^{K} \lambda_j - \epsilon \sum_{j=1}^{K} \sum_{l=1}^{N_c} f\left(\frac{\lambda_j - M_l}{\epsilon}\right) + \epsilon \sum_{j=1}^{K} \sum_{l=1}^{N_c} f\left(\frac{\lambda_j - \tilde{M}_l}{\epsilon}\right)$$

$$+ \sum_{i,j=1}^{K} f\left(\frac{\lambda_i - \lambda_j - \epsilon}{\epsilon}\right) \qquad \left(f(x) = x(\log x - 1)\right)$$

The minima / Vacuum is given by $\frac{\partial W_{eff}(\lambda)}{\partial \lambda} = 0$, explicitly

$$\prod_{l=1}^{N_c} \frac{\lambda_j - M_l}{\lambda_j - \tilde{M}_l} = q \prod_{k \neq j}^{K} \frac{\lambda_j - \lambda_k - \epsilon}{\lambda_j - \lambda_k + \epsilon}, \quad q = e^{2\pi i \tau}(-1)^{K+1}$$

This looks Strikingly Close to "Bethe Ansatz Equation" to "inhomogeneous twisted XXX Spin Chain"! (indeed the case after identifying parameters)

(Obtained from Algebraic Bethe Asatze)

This is an example that "Vacuum" of gauge theory is encoded in a "Quantum Integrable System"

We have seen similar "moduli Space / Integrable System Connection", from the "Coulomb branch of 4dim $\mathcal{N}=2$ SUSY gauge theories" (Seiberg - Witten)

In fact, the quantum spin chain we just met, its classical cousin has already made appearance in the following 4dim Gauge Theory

"$\mathcal{N}=2$ Super-QCD"

We have $U(N_c)$ Vector Multiplet +
- N_c "fundamental hypermultiplets" of masses $(m_1, m_2 \quad m_{N_c})$
- N_c "Anti-fundamental hypermultiplets" of masses $(\tilde{m}_1, \tilde{m}_2, \cdots \tilde{m}_{N_c})$

The theory also has Complex gauge coupling $\tau = \frac{4\pi i}{g_{4D}^2} + \frac{\theta}{2\pi}$

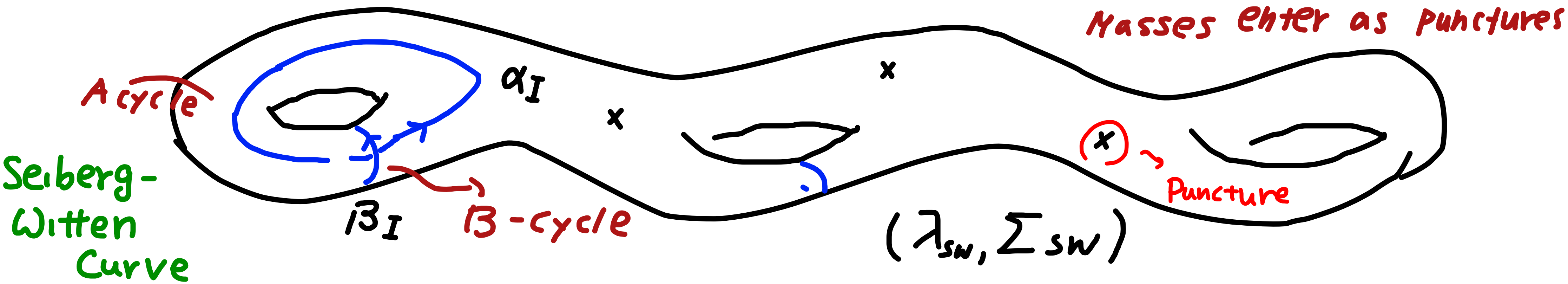
The moduli Space has "Coulomb" and "Higgs Branch"

Higgs Branch $\langle squark \rangle \neq 0$, protected from quantum corrections by non-renormalizations

Coulomb Branch $\quad [\phi, \phi^\dagger]^2 = 0 \quad$ ($\phi \sim$ Adjoint Scalar in Vector Multiplet)

$\langle \phi \rangle \sim \mathrm{diag}(\phi_1, \phi_2 \;\; , \phi_{N_c})$, $U(\underbrace{N_c) \longrightarrow U}_{\text{Generalized EM}}(1)^{N_c}$ $\quad (\Lambda \ll \langle \phi \rangle)$

- Classical Coulomb metric can be "quantum Corrected", but only one-loop perturbatively, it can further receive non-perturbative instanton Corrections (Difficult to compute order by order)
- The famous work of Seiberg and Witten, determined exactly the quantum Coulomb branch metric / effective Coupling
- They do so by relating the system to an auxilary algebraic curve



The gauge Coupling τ is the "Complex Structure" of Σ_{SW}

or the "ratio" between A & B cycles

Holomorphic 1-form

The integrals of λ_{SW} over A & B cycles play the role of

"Electric" and "Magnetic" Coordinates in low energy SQED

(Entering KE)

$$a_I(u) = \oint_{\alpha_I} \lambda_{SW}(u) \quad , \quad a_I^D(u) = \oint_{\beta_I} \lambda_{SW}(u)$$

Related by $\mathcal{F}(\vec{a}, \vec{m})$ "Prepotential"

(u ~ moduli on the Curve, Coming from Weyl invariant Combination of $\langle \phi \rangle$)

<u>Example</u>

"Pure $\mathcal{N}=2$ $SU(N_c)$ gauge theory"

$$\Sigma_{SW} \qquad \omega + \frac{\Lambda^{2N_c}}{\omega} = P_{N_c}(x, u) \quad , \quad P_{N_c}(x,u) = \det(x \mathbb{1} - \langle \phi \rangle)$$

$$= x^{N_c} - \sum_{k=0}^{N_c - 1} u_k x^k$$

$$\lambda_{SW}(u) = \frac{1}{2\sqrt{2}\pi} x(\omega, u) \frac{d\omega}{\omega}$$

- But we have also seen identical algebraic curve before, this is precisely the so-called "Spectral Curve" for A_{N-1} Toda-System (Classical)

- Or for our prototype $\mathcal{N}=2$ Super-QCD, the Seiberg-Witten curve Σ_{SW} is given by (Argyres, Plesser, Shapere)

$$\Sigma_{SW} \qquad \omega^2 \prod_{\ell=1}^{N_c}(x-\tilde{m}_\ell) - 2\omega \prod_{\ell=1}^{N_c}(x-a_\ell) - h(h+2)\prod_{\ell=1}^{N_c}(x-m_\ell)=0$$

$$h = -\frac{2q}{1+q}, \quad q = e^{2\pi i \tau} \sim \text{Instanton correction}$$

Again, we have seen similar algebraic curve in "Classical integrable system", this is the spectral curve for "inhomogeneous twisted XXX spin chain"

$$P(z) = 2\prod_{\ell=1}^{N_c}(z-\phi_\ell)$$

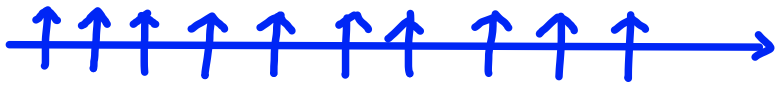
$$\Sigma_{XXX} \qquad t^2 - 2P(z)t + h(h+2)K_+(z)K_-(z) = 0$$

$$K_\pm(z) = \prod_{\ell=1}^{N_c}(z-\theta_\ell \pm i J_\ell)$$

Indeed , we can identify $(a_I(u), a_I^D(u))$ with Canonical Coord from H_{XXX}, also $\langle \mathrm{Tr}\,\phi^k \rangle$ with Commuting Conserved Charges.

- So far we has discussed how 4dim SUSY gauge theories can be related to Classical Integrable Systems, but can we also see "Quantum Integrable Systems" emerge from 4 dim theories ?

- To quantize a dynamical system, we need to identify Appropriate Vacuum, in our XXX spin chain, we can start with "Ferromagnetic Vacuum"



- In SQCD, after we identify the parameters in gauge theory and integrable spin chain, this precisely corresponds to

"Root of Baryonic Higgs" $\quad a_\ell - m_\ell = 0 \;, \quad \ell = 1, \quad , N_c$

Q. What about "Planck Constant $\hbar$" for quantization?

A. Nekrasov and Shatashvili proposed a striking idea, that is to put 4dim $\mathcal{N}=2$ gauge theories on "Curved Background", or a "topologically twisted version" (Coupling spacetime with internal symmetries)

More Concretely, We consider the Euclidean Rotational Group

$\mathbb{R}^4 \simeq \mathbb{C}\times\mathbb{C}$ $\quad U(1)_1\times U(1)_2 \subset SO(4)\simeq SU(2)_1\times SU(2)_2$

and break the rotational symmetry in two out of four directions, with the deformation parameter "ϵ" $\left((z_1, z_2)\to(z_1, e^{i2\pi\epsilon}z_2)\right)$

- Both Rotational Symmetry and Supersymmetry are partially Broken. Only preserved in x^1-x^2 plane, $\mathcal{N}=(2,2)$ in 2 dimensions. "Notice that this is however different $\mathcal{N}=(2,2)$ theory from earlier discussion"

- The proposal of Nekrasov-Shatashvili is that, in such deformed $\mathcal{N}=2$ theory, if we compute the prepotential $\mathcal{F}(\vec{a},\vec{m},\epsilon)$

("Kinetic term $\sim \mathrm{Im}\left(\frac{\partial \mathcal{F}}{\partial a}\bar{a}\right)$, gauge coupling $\tau \sim \partial_a^2 \mathcal{F}$)

- Proposal $\mathcal{F}(\vec{a},\vec{m},\epsilon)$ is the Yang-Yang functional of the for quantizing the underlying classical integrable system That is

$$\frac{1}{2\pi i}\frac{\partial \mathcal{F}(\vec{a},\vec{m},\epsilon)}{\partial \vec{a}} = \vec{n}\epsilon \quad \rightsquigarrow \epsilon \text{ Planck constant } \hbar$$

- From perspective of Σ_{SW}, $\frac{\partial \mathcal{F}}{\partial a}$ ($\rightsquigarrow a_D$) can be identified with the B-cycle / dual cycle of $\vec{a}$ $\rightsquigarrow$ We are quantizing "Canonical Momentum"
- Many possible choices of Quantizations, they are related via "Electromagnetic Duality Transformation"

- After long detour in 4 dim, let us recall in the original 2-dim theory, the BAE arises from "Standard quantization condition" in algebraic ansatz equation. That is "quantizing around Ferromagnetic Vacua"

$$\vec{a} - \vec{m} = \vec{n}\,\epsilon$$ ~ identification of $\hbar \sim \epsilon$

- Proposal (Chen et al, also CDL)

Partition of flux tubes

$$\mathcal{F}(\vec{a}=\vec{m}-\vec{n}\epsilon) - \mathcal{F}(\vec{a}=\vec{m}-\epsilon\vec{1}) = \mathcal{W}(\vec{n})$$

vacuum-dependent

Proved in (Chen et al)

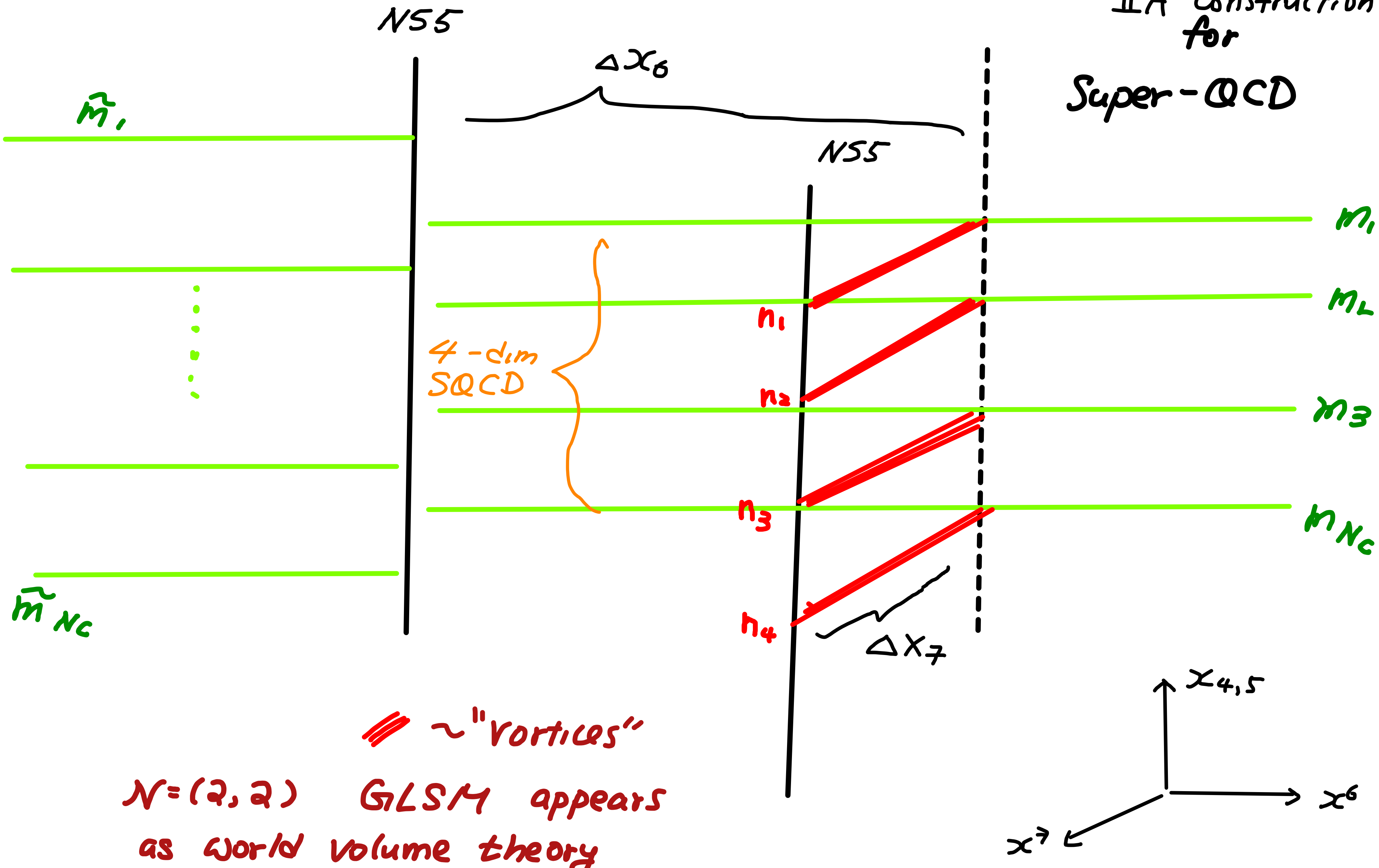
Can be argued from both "Identical Quantization Condition"

+

The minima obtained from the potentials in 2 dim and 4 dim Gauge theories both coincide with eigenstates of $\mathcal{H}_{XXX}$

Brane Construction for 2d/4d Duality

"IIA Construction" for Super-QCD



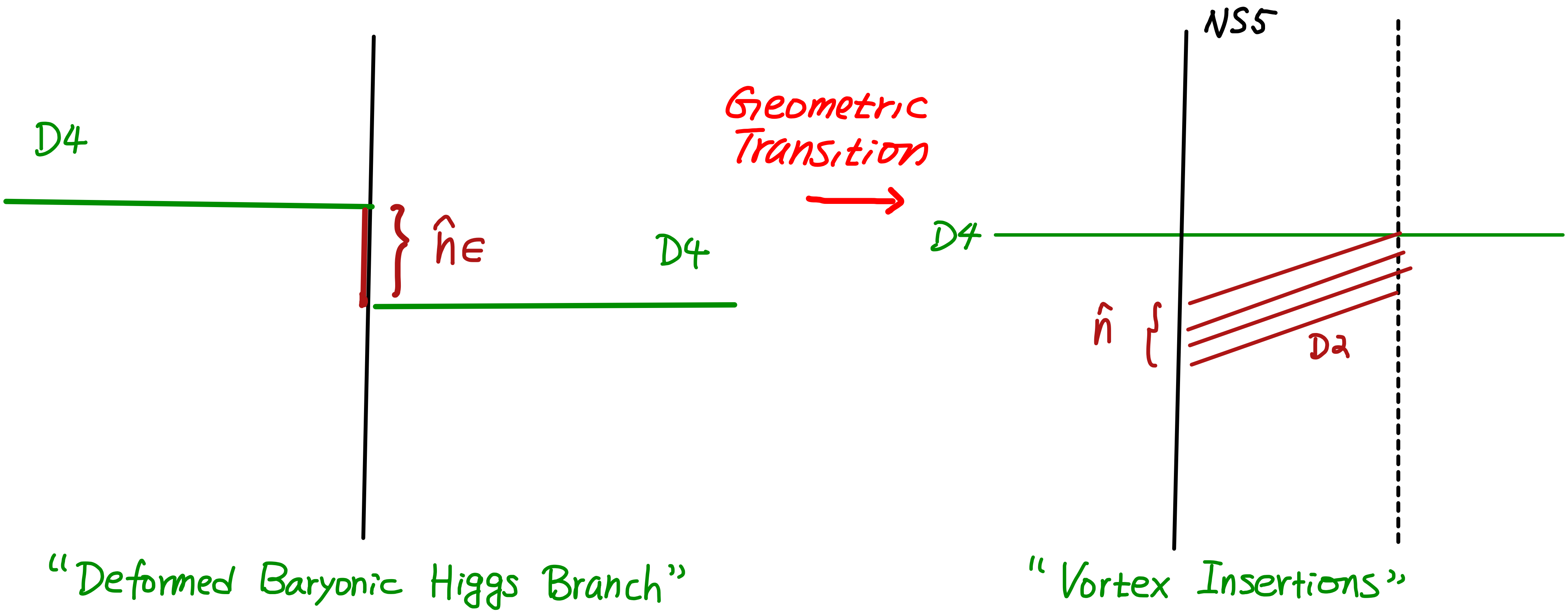
$\mathcal{N}=(2,2)$ GLSM appears as world volume theory

- More Intuitive Explanation (HYC + Annamaria Sinkovics)

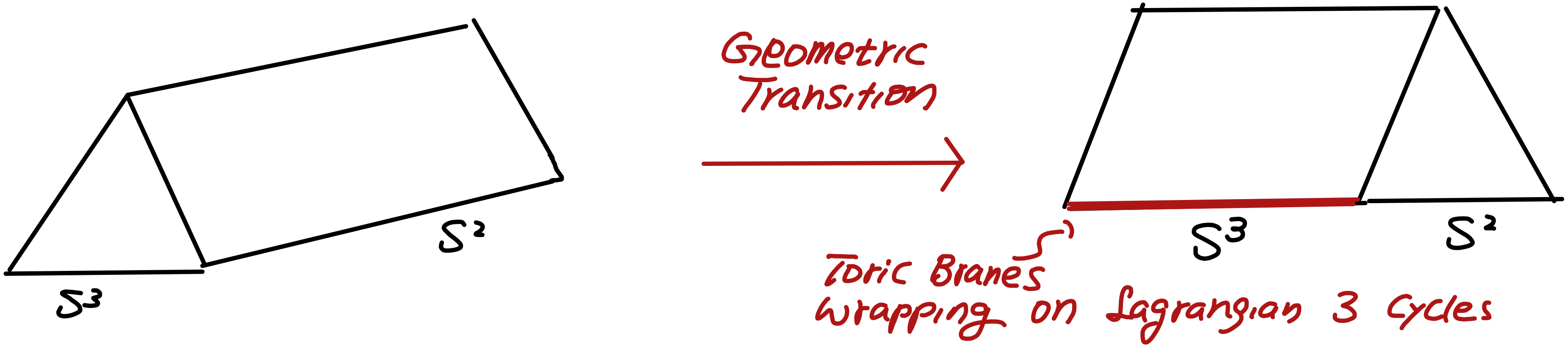
"2d SUSY GLSM is the Vortex World Volume Theory of the dual 4d SUSY Theory"

Probing Specific 4d Vacua Baryonic Higgs Root

D-brane Pictures



This is in fact a realization of "Geometric Transition"



"Resolved Conifold" "Deformed Conifold"

This picture allows to construct many other "Duality Pairs", connecting Geometric Transition with Integrable Structures in Supersymmetric Gauge Theories (HYC + Annamaria Sinkovics)